An empirical comparison of the expressiveness of the additive value function and the Choquet integral models for representing rankings

M. Pirlot, H. Schmitz
UMONS Université de Mons, Faculté Polytechnique, Belgium
marc.pirlot@umons.ac.be, helene_stz@hotmail.com

P. Meyer, Telecom Bretagne, France
patrick.meyer@telecom-bretagne.eu

ABSTRACT.
Recent – and less recent – work has been devoted to learning additive value functions or a Choquet capacity to represent the preference of a decision maker on a set of alternatives described by their performance on the relevant attributes. In this work we compare the ability of related models to represent rankings of such alternatives. Our experiments are designed as follows. We generate a number of alternatives by drawing at random a vector of evaluations for each of them. We then draw a random order on these alternatives and we examine whether this order is representable by a simple weighted sum, a Choquet integral with respect to a 2- or 3-additive capacity, an additive value function in general or a piecewise-linear additive value function with 2 or 3 pieces. This analysis is performed using linear programming and the Kappalab R package. The results explore how representability depends on varying the numbers of alternatives and criteria.

KEYWORDS. Preference representation, additive value function, Choquet integral, weighted sum.

1. INTRODUCTION

Additive value functions occupy a dominant position in the models used for representing the preferences of a decision maker (DM) on alternatives described by their performance on several attributes (Keeney, Raiffa, 1976, Bouyssou and Pirlot, 2004). This model may be used under the necessary condition that the DM's preference satisfies the preference independence condition (Keeney and Raiffa, 1976, p.32). In the late 1980's another model for representing preferences in the multi-attribute context has emerged. The use of the Choquet integral – which is better known in the context of decision under risk (Schmeidler, 1986) – is advocated in the multi-attribute case when a form of interaction between criteria is presumed (Grabisch, 1996). A simple example is that of the selection process of students applying for graduate studies in management (Grabisch and Labreuche, 2004). Students are evaluated by their past performance in mathematics, statistics and language skills. Since skills in maths and stats are correlated (maths and stats are, to some extent, redundant attributes), for students good at maths, the jury responsible for the selection prefers a student with good linguistic skills to one that is good at stats. Things will go the other way around for students that have a weakness in mathematics. In such a situation, there is a (negative) interaction between criteria maths and stats and a break of preference independence. Note however that interaction between criteria is a property that should not be identified with the fact that the preference independence hypothesis is violated: preferences that can be represented by a Choquet integral do not necessarily satisfy the preference independence condition, but they do satisfy weaker forms of independence such as comonotonic independence (see Wakker, 1989, p. 111) and weak separability (see Bouyssou and Pirlot, 2004).

The goal of the present work is to study the ability of the Choquet integral at representing preferences as compared to that of the traditional additive value function model. This is the "expressiveness" of the model. In this study, we assume that the Choquet integral is computed directly on the vector of performances of the alternatives, without any recoding of these performances by marginal value functions. The additive value function model, on the contrary, takes the preferences of the DM on each attribute into account by modeling them using possibly non linear functions defined on the range of values that is relevant for each attribute. In this setting, there obviously are examples in which the preference can be represented by means of an additive value function but not using a Choquet integral. One may also conceive of a model in which the Choquet integral is applied to a vector of marginal value functions. Such a model would be more expressive since it would encompass the additive value function model. We briefly explain in section 4 how it is possible to get an indication on how much more expressive is such a model as compared to the simple additive value function model involving the same marginal value functions.

In order to assess the relative expressivity of these models, we have performed series of computational experiments which we shall describe in section 2. We then analyze the results, draw some conclusions and suggest further investigations that seem interesting.

2. THE EXPERIMENTAL SETTING

The experimental design is roughly the following. For given numbers *na* of alternatives and *nc* of criteria (or attributes), we generate *na* alternatives, i.e. *na* vectors having *nc* components. The values of the *nc* components can be seen as the performances of the corresponding alternative on the *nc* criteria (or attributes). These are randomly drawn from the uniform distribution on the [0,1] interval. We assume w.l.o.g. that the decision maker prefers larger values to smaller ones on all attributes. To fix the ideas, consider an alternative *a*; we denote by $(a_1, \ldots, a_{nc})$ the performance vector associated with alternative *a*.

An strict total order on the *na* alternatives is drawn at random from a uniform distribution on all orders. We then use linear programming or appropriate other routines for checking whether this random order on the alternatives can be represented in various models. In each of the considered models, a score is computed for each alternative, aggregating the performances on the various attributes. Alternative *a* is judged to be preferred to *b* if the score of *a* is greater than the score of *b*. The following models have been considered:

1. The *weighted sum* for which the score of alternative *a* is computed as $\sum_{i=1}^{nc} w_i a_i$, where w_i are weights that need be determined
2. The *additive value function* for which the score of alternative *a* is computed as $\sum_{i=1}^{nc} u_i(a_i)$, where u_i are (marginal value) functions from the [0,1] interval into itself that need be determined. For some experiments, two special cases of the general model will also be considered, in which u_i is a piecewise linear marginal value function. In the first variant, u_i has two linear pieces corresponding to a division of the [0,1] interval in two equal parts. In the second variant there are three linear pieces, the [0,1] interval being divided in three equal sub-intervals. These two variants will only appear in Figure 2.
3. A *Choquet integral* in which the score of alternative *a* is computed using a particular capacity; 2-*additive* and 3-*additive* capacities are used. In the case of a 2-additive capacity, the score of *a* is computed by means of the following formula: $\sum_{i=1}^{nc} m_i a_i + \sum_{i,j,i<j} m_{ij}(a_i \wedge a_j)$, where m_i (resp. m_{ij}) are weights associated to the criteria (resp. the pairs of criteria) and $a_i \wedge a_j$ denotes the minimum of a_i and a_j. There are some constraints on the weights which we do not explicitly state here (see e.g. Grabisch et al., 2008). The formula for 3-additive capacities is similar; it involves an additional term with weights m_{ijk} associated with triplets of criteria.

All the experiments, including the generation of the alternatives and the orders, are performed using R, a free software environment for statistical computing and graphics.

The parameters of the first two models are searched for using a linear programming software (lpsolve) that can be called from R. The linear program involving the constraints expressing the order on the alternatives that has been randomly generated is submitted to the solver. The structure of the constraints for piecewise linear marginal value functions are inspired by the aggregation-disaggregation approach UTA (Jacquet-Lagrèze and Siskos, 1982), while the formulation for checking the existence of an additive value function without any restriction on the type of marginal value functions allowed (except that they are non-decreasing) is taken from (Greco et al., 2008). For checking the representability by means of a Choquet integral and a 2- or 3-additive capacity, we use the Kappalab R package and, specifically, its lin.prog.capa.ident command. In all models, the optimization routine seeks to maximize variable δ, the minimal difference in score of two alternatives that are ranked consecutively in the random order.

3. THE RESULTS

We systematically explored the cases of a number of alternatives *na* = 4, 5, 6, 10 and, for each value of *na*, a number of criteria *nc* = 3, 4, 5, 6, 8. For small *na* (4 to 6), we generated between 50 and 100 instances of *na* alternatives and we systematically tried to represent all orders on the alternatives. For *na* = 8 and 10, we generated between 25 and 50 instances and considered only a random sample of 3000 orders. Table 1 displays the percentage of the orders that can be represented by the various models.

Table 1. Expressivity of the models (%)

na	nc	Tot Nb	Inst	Orders	Add	3-cap	2-cap	WSum
4	3	2400	100	24	58.1	48.6	48.6	34.0
4	4	2400	100	24	72.7	69.1	67.7	53.8
4	6	2400	100	24	90.6	87.5	87.3	75.0
4	8	2400	100	24	96.5	-	95.6	91.2
6	3	36000	50	720	24.0	15.0	13.5	4.8
6	4	43920	61	720	45.0	31.7	28.4	11.8
6	6	36000	50	720	78.9	69.4	66.1	38.7
6	8	36000	50	720	96.0	-	93.2	64.3
8	4	75000	25	3000	25.0	6.9	5.3	0.9
8	6	90000	30	3000	74.9	56.4	44.5	7.9
8	8	90000	30	3000	96.2	-	74.7	23.3
10	4	90000	30	3000	17.84	2.8	0.05	0.00
10	6	120000	40	3000	61.4	26.5	15.7	0.01
10	8	150000	50	3000	81.4	-	39.1	3.8

In Table 1, “Add” stands for “additive value function” (column 6), “3-cap” for “3-additive capacity” (column 7), “2-cap” for “2-additive capacity” (column 8) and “WSum” for “weighted sum” (column 9). The 3rd column shows the total number of trials, the 4th, the number of instances, the 5th, the number of orders. We display the results only for *na* = 4, 6; *nc* = 3, 4, 6, 8 and *na* = 8, 10; *nc* = 4, 6, 8. The results for 3-additive capacities could never be obtained when *nc* = 8 (error message issued by R probably for lack of memory space on the computer used for the experiments).

The following conclusions can be drawn from the percentages shown in Table 1:

1. for all methods, increasing the number of criteria improves expressivity; in contrast, increasing the number of alternatives reduces expressivity
2. the additive value function model is more expressive than the Choquet integral with 3-additive capacity; the latter is slightly more expressive than Choquet integral with 2-additive capacity; finally, this method is more expressive than the weighted sum. These differences are all the more marked that *na* is large
3. 10 alternatives described by 6 criteria is very seldom representable by a weighted sum while additive value functions can represent it in 60% of the cases.

Table 2. Comparing expressivity (%)

na	nc	TotNb	Inst	Orders	Add/ 3cap	Add/ 2cap	3cap/ 2cap	2cap/ WSum	2cap/ Add	3cap/ Add
4	3	2400	100	24	9,5	9,5	0,0	14,7	0,9	0,9
4	4	2400	100	24	3,6	5,0	1,4	13,9	0,0	0,0
4	6	2400	100	24	3,1	3,3	0,2	12,5	0,0	0,0
4	8	2400	100	24	-	0,9	0,0	4,3	0,0	-
6	3	36000	50	720	13,2	16,6	3,4	16,6	0,2	0,2
6	4	43920	61	720	13,2	16,6	3,4	16,6	0,2	0,2
6	6	36000	50	720	9,5	12,8	3,3	27,4	0,04	0,07
6	8	36000	50	720	-	2,8	-	29,0	0,0	-
8	4	75000	25	3000	18,1	19,6	1,5	4,4	0,4	0.4
8	6	90000	30	3000	18,7	30,5	11,9	36,6	0,15	0,05
8	8	90000	30	3000	-	21,6	-	51,3	0,05	-
10	4	90000	30	3000	15,1	17,8	2,7	0,0	0,0	0,3
10	6	50000	50	1000	29,7	40,0	10,8	15,7	0,3	0,0
10	8	150000	50	3000	-	42,3	-	35,3	0,4	-

The figures in Table 1 do not tell us the percentages of the cases in which a representation by a Choquet integral with 2- or 3-additive capacities exists while no additive value function can fit the data, and the other way around. Such results are displayed in Table 2. For instance, column labeled “Add/3cap” yields the percentage of orders that can be represented by an additive value function but cannot be represented using a Choquet integral with 3-additive capacities; column “3cap/Add” describes the opposite case; 3cap/2cap assesses the improvement generated by passing from 2- to 3-additive capacities. Let us explain, for example, how to read the first row in table 2. One hundred instances (Inst) of 4 alternatives (na) evaluated on 3 criteria (nc) have been generated. All possible orders on these alternatives (Orders = 24) have been considered, which makes a total number of 2400 trials (TotNb = 2400). Among them, 9,5% can be represented by an additive value function but not by a Choquet integral with 3-additive capacity (Add/3cap) or 2-additive capacity (Add/2cap). Using a 3-additive capacity does not enable to represent any of the orders that cannot be represented by a Choquet model with 2-additive capacity (3cap/2cap = 0%). 14,7% of the orders that cannot be represented by a weighted sum can be represented by a Choquet model with 2-additive capacity (2cap/WSum) but this is only the case for 0,9% of the orders that cannot be represented by an additive value function (2cap/Add). Using 3-addititive capacities instead of 2-additive ones does not improve this figure (3cap/Add).

From the figures in Table 2, we see that there are very few cases (less than 1%) in which 2- or 3- additive capacities can represent an order that additive value functions cannot. The advantage of using 3- instead of 2-additive capacities is not striking up to *na* = 6. For *na* = 8 (resp. 10), the gain of using a 3-additive capacity reaches 11,9% (resp. 10,8%). This would probably be all the more true for *nc* = 8 but it becomes computationally difficult to find a 3-additive capacity for problems of this size.

Comparing the expressivity of the Choquet integral with that of a weighted sum, we see that the percentage grows from about 12% for *na* = 4 up to around 25% for *na* = 6; for *na* = 8 and 10, the percentage is extremely small for 4 criteria and rapidly grows with the number of criteria.

The conclusions drawn form Tables 1 and 2 can be confirmed in the following way. In the linear programming formulation of the representation problem as well as in the search for a capacity, we have used the minimal score difference δ between consecutive alternatives in the ranking as an objective function. So the larger this difference, the easier the model fits the data. Figures 1 and 2 below represent the histogram of the values of δ obtained in the different models.

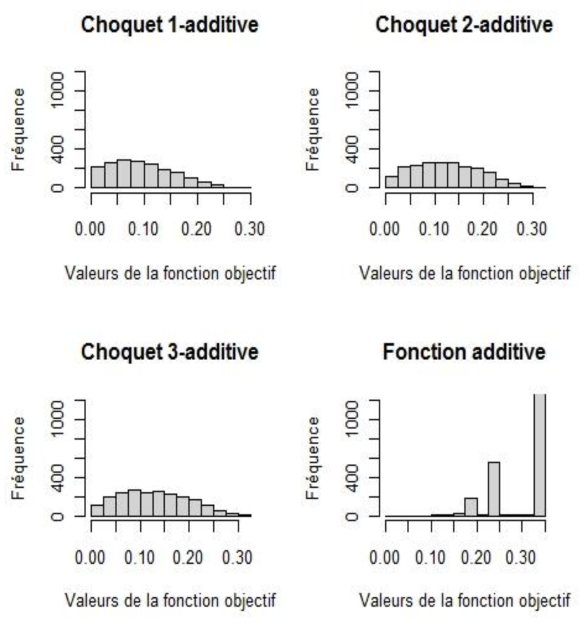

Figure 1. Values of δ: Choquet vs. Additive

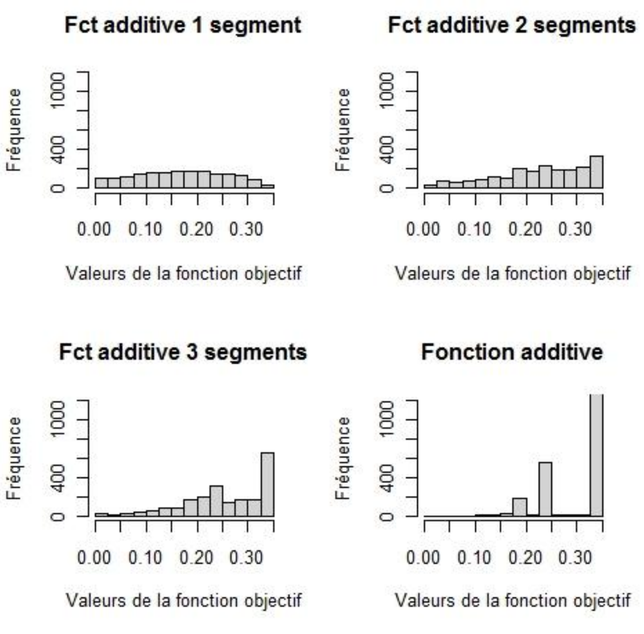

Figure 2. Values of δ: Variants of Additive

Figure 1 shows that large values of δ are much more frequent for additive value function representations. Figure 2 shows that for two segments or three segments value functions there is already a tendency to obtain larger values of δ than with the Choquet integral. Note that additive value functions "with one segment" correspond to the weighted sum and serve as reference point.

Expressivity as a function of the Kendall distance w.r.t. a reference order.

We have also studied the expressivity of the models as a function of the distance of the random orders from a reference order. The experimental design is the following. We generate the alternatives and the orders as above. For a given set of alternative we compute the sum of the evaluations of each alternative (unweighted sum) and we rank the alternatives according to that score. The corresponding ranking serves as a reference order. For each random order that is generated, we compute its Kendall distance to the reference order (i.e., essentially, the number of pairs in the symmetric difference of the two orders). We sort the generated orders in categories of Kendall distances from the reference order. Three experiments have been performed, for 5 (resp. 10, 20) alternatives and 4 (resp. 6, 8) criteria. In each experiment, 100 sets of alternatives are randomly generated and all permutations (120) for the case $na = 5$; in the cases of $na = 10$ and 20, 300 permutations have been drawn at random. What is observed is a *slow* decrease of the expressivity of all methods for increasing Kendall distances.

4. CONCLUSIONS AND PERSPECTIVES

The additive value function model appears to be definitely more expressive than the Choquet integral (using tractable 2- or 3-additive capacities) at least for the ranges of numbers of alternatives and criteria that we have explored. These ranges are rather typical in the applications of methods for learning preferences such as UTA (Jacquet-Lagrèze and Siskos, 1982) or UTA-GMS (Greco et al., 2008). It cannot be excluded that our conclusions could be challenged in applications with larger learning sets. However, we have made exploratory trials with 20 alternatives and 8 criteria (using additive value functions, Choquet with 2-additive capacities and weighted sum) resulting in outcomes that are in line with our previous conclusions.

If we now compare the Choquet integral with the weighted sum, we see that using Choquet with 2- or 3-additive capacities may increase significantly the percentage of representable orders (around 25% on average, for 2-additive capacities). The gain of expressivity w.r.t. the weighted sum seems to be maximal when *na* is approximately equal to *nc* (but this requires further confirmation).

The Choquet integral is sometimes used instead of a sum in the additive value function model (Grabisch et al. 2004), obviously increasing the expressivity of the latter. An indication on how much more expressive is such a model as compared with the simple additive value function model can be obtained by comparing the expressivity of the Choquet integral with respect to the weighted sum. Indeed, assuming that the distribution of the values of marginal value functions is uniform, we may interpret our randomly generated vectors as representing the *marginal values* of the alternatives instead of just performances. Under this hypothesis, a weighted sum of these marginal values can be interpreted as an additive value function. Hence, the results of the previous paragraph may be considered as an estimate of what can be gained in terms of expressivity by considering a Choquet integral instead of a sum in the additive value function model.

In view of these results, one might be tempted to recommend to restrict the use of a Choquet integral to the cases in which marginal value functions have already been elicited and there is strong evidence of interaction between criteria (since the notion of interaction is far from being clear when evaluations have not been recoded into marginal values or at least when the scales of the various criteria have not been made commensurate; see Grabisch and Labreuche, 2004, on the notion of commensurateness). We might be less restrictive however if we consider that the most expressive models are not necessarily the best choice in learning models, since they generally involve the specification of many parameters. Indeed, if information about the DM’s preferences is scarce, using a less expressive model, such as a 2-additive Choquet integral, may be advocated since it will generally lead to a lower degree of indetermination of the parameters than with an additive utility function. This is all the more true that the DM or the analyst has the intuition that the criteria do interact (although, again, such an “intuition” should be considered with a dose of critical sense). Note that an alternative to general additive value functions is using piecewise linear additive value functions as is done in UTA for instance (Jacquet-Lagrèze and Siskos, 1982); such models require the elicitation of fewer parameters than the general additive value function model.

The empirical study accounted for above is by no means the final word on these questions. Many interesting points remain to be explored, among which we would like to mention:

- for a fixed number of criteria, study the impact of the size of the learning set (consisting of randomly generated alternatives) on the possibility of representing the DM's preferences by an additive value function or a Choquet integral. More precisely, the experiment could be done as follows: choose a maximal value for *na*, for instance *na* = 100; generate *na* alternatives and a random order on them; select more and more alternatives from the set of *na* generated alternatives and see whether the ranking induced on the subset is representable by a given model; stop when it is no longer representable and iterate the process with another random order; iterate with another set of alternatives
- study the frequency with which randomly generated sets of alternatives and orders are not compatible with the preference independence assumption; study the frequency of dominated alternatives, i.e. alternatives that are worse on all criteria than another alternative in a randomly generated set of alternatives
- try to better understand the notion of interaction between criteria; this could be done for instance by studying the orders that are representable by a Choquet integral but not by an additive value function.

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